

# KÜNNETH FORMULA IN RABINOWITZ FLOER HOMOLOGY

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**ABSTRACT.** Rabinowitz Floer homology has been investigated on submanifolds of contact type. The contact condition, however, is quite restrictive. For example, a product of contact hypersurfaces is rarely of contact type. In this article, we study Rabinowitz Floer homology for product manifolds which are not necessarily of contact type. We show for a class of product manifolds that there are infinitely many leafwise intersection points by proving the Künneth formula for Rabinowitz Floer homology.

## 1. INTRODUCTION

Rabinowitz Floer homology has been extensively studied in recent years because of its interrelation with the leafwise intersection problem. However Rabinowitz Floer homology (to be honest, the perturbed Rabinowitz action functional) has worked principally on a contact submanifold and little research has been conducted on a non-contact case. Our primary objective in this paper is to find leafwise intersection points and define Rabinowitz Floer homology for this class of submanifolds which are not necessarily of contact type. In addition we show for the class that there are infinitely many leafwise intersection points by proving the Künneth formula for Rabinowitz Floer homology. For simplicity, throughout this paper, we use  $\mathbb{Z}/2$ -coefficients for Rabinowitz Floer homology, but expect the Künneth formula continues to hold with  $\mathbb{Z}$ -coefficient.

We consider restricted contact hypersurfaces  $(\Sigma_1, \lambda_1)$  resp.  $(\Sigma_2, \lambda_2)$  in exact symplectic manifolds  $(M_1, \omega_1 = d\lambda_1)$  resp.  $(M_2, \omega_2 = d\lambda_2)$ . Moreover we assume that  $\Sigma_1$  resp.  $\Sigma_2$  bounds a compact region in  $M_1$  resp.  $M_2$  and that those  $M_1$  and  $M_2$  are convex at infinity; that is, they are symplectomorphic to the symplectization of a compact contact manifold at infinity. Given  $F_1 \in (S^1 \times M_1)$ ,  $F_2 \in C^\infty(S^1 \times M_2)$ , the operation

$$(F_1 \oplus F_2)(t, x, y) = F_1(t, x) + F_2(t, y), \quad (t, x, y) \in S^1 \times M_1 \times M_2$$

provides a time-dependent Hamiltonian function  $F_1 \oplus F_2 \in C^\infty(S^1 \times M_1 \times M_2)$ . We also introduce projection maps  $\pi_1 : M_1 \times M_2 \rightarrow M_1$  and  $\pi_2 : M_1 \times M_2 \rightarrow M_2$ ; then  $(M_1 \times M_2, \omega_1 \oplus \omega_2)$  admits the symplectic structure  $\omega_1 \oplus \omega_2 = \pi_1^* \omega_1 + \pi_2^* \omega_2$ .

On  $(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$ , we define the perturbed Rabinowitz action functional  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  as in (2.1). Since  $\Sigma_1 \times \Sigma_2$  is a stable submanifold, we can define Floer homology of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  when  $F \equiv 0$  (refer to [24] for definitions and constructions). This Floer homology  $\text{HF}(\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2})$  is called *Rabinowitz Floer homology* and denoted by  $\text{RFH}(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$ , see Section 3. By the standard continuation method in Floer theory,  $\text{HF}(\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2})$  and  $\text{RFH}(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$  are isomorphic whenever  $\text{HF}(\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2})$  is defined.

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**Theorem A.** *The Floer homologies  $\mathrm{RFH}(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$  and  $\mathrm{HF}(\mathcal{A}_{F_1 \oplus F_2}^{\tilde{H}_1, \tilde{H}_2})$  are well-defined. Moreover, we have the following Künneth formula in Rabinowitz Floer homology:*

$$\mathrm{RFH}_n(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \cong \bigoplus_{p=0}^n \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{RFH}_{n-p}(\Sigma_2, M_2).$$

Here,  $\mathrm{RFH}_p(\Sigma_1, M_1)$  (resp.  $\mathrm{RFH}_{n-p}(\Sigma_2, M_2)$ ) is the Rabinowitz Floer homology for the restricted contact hypersurface  $\Sigma_1$  in  $M_1$  (resp.  $\Sigma_2$  in  $M_2$ ), see [1] or Section 3.

**Remark 1.1.** In this paper, we unfortunately establish compactness of gradient flow lines of the Rabinowitz action functional only for perturbations of the form  $F = F_1 \oplus F_2$ . Thus we cannot study the existence problem of leafwise intersection points for an arbitrary perturbation. However, if  $\Sigma_1 \times \Sigma_2$  has contact type in the sense of Bolle [10, 11] (see Section 4), the Floer homology  $\mathrm{HF}(\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2})$  is defined for all perturbations, see [24]. We note that, in general,  $\Sigma_1 \times \Sigma_2$  is not of contact type in the sense of Bolle. For example,  $S^3 \times S^3$  is not a contact submanifold in  $\mathbb{R}^8$ , see Remark 4.2.

**Question 1.2.** What perturbations have a leafwise intersection point on  $(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$ ?

**Remark 1.3.** Once one verifies compactness of gradient flow lines of the Rabinowitz action functional for a given perturbation  $F$ , it guarantees the existence of leafwise intersection points for that  $F$  by using the stretching the neck argument in [1]. In this paper, we are able to compactify gradient flow lines of  $\mathcal{A}_{F_1 \oplus F_2}^{\tilde{H}_1, \tilde{H}_2}$ , and thus it guarantees the existence of leafwise intersection points of  $F_1 \oplus F_2$ ; but, this directly follows from the result in [1] that each  $F_1$  and  $F_2$  has a leafwise intersection point on  $\Sigma_1$  and  $\Sigma_2$  respectively.

**Definition 1.4.** The *Hamiltonian vector field*  $X_F$  on a symplectic manifold  $(M, \omega)$  is defined explicitly by  $i_{X_F}\omega = dF$  for a Hamiltonian function  $F \in C^\infty(S^1 \times M)$ , and we call its time one flow  $\phi_F$  the *Hamiltonian diffeomorphism*. We denote by  $\mathrm{Ham}_c(M, \omega)$  the group of Hamiltonian diffeomorphisms generated by compactly supported Hamiltonian function. This group has a well-known norm introduced by Hofer (see Definition 2.2).

**Definition 1.5.** We denote by  $\wp(\Sigma_1, \lambda_1) > 0$  the *minimal period* of closed Reeb orbits of  $(\Sigma_1, \lambda_1)$  which are contractible in  $M_1$ . If there is no contractible closed Reeb orbit we set  $\wp(\Sigma_1, \lambda_1) = \infty$ .

In Theorem B we do not consider  $\Sigma_2$ , and  $M_2$  need to be closed.

**Theorem B.** *Let  $(M_2, \omega_2)$  be a closed and symplectically aspherical, i.e.  $\omega_2|_{\pi_2(M_2)} = 0$ , symplectic manifold. Then, although  $\Sigma_1 \times M_2$  is not a contact hypersurface,*

- (B1)  *$\Sigma_1 \times M_2$  has a leafwise intersection point for  $\phi \in \mathrm{Ham}_c(M_1 \times M_2, \omega_1 \oplus \omega_2)$  with Hofer-norm  $\|\phi\| < \wp(\Sigma_1, \lambda_1)$  even if  $\Sigma_1$  does not bound a compact region in  $M_1$ .*
- (B2) *The Rabinowitz Floer homology  $\mathrm{RFH}(\Sigma_1 \times M_2, M_1 \times M_2)$  can be defined when  $\Sigma_1$  bounds a compact region in  $M_1$ . Moreover, we have the Künneth formula:*

$$\mathrm{RFH}_n(\Sigma_1 \times M_2, M_1 \times M_2) \cong \bigoplus_{p=0}^n \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{H}_{n-p}(M_2).$$

To prove Theorem B without any contact conditions, we need to show a special version of isoperimetric inequality, see Lemma (3.2).

**Remark 1.6.** It is worth emphasizing that  $\Sigma_1 \times M_2$  is not necessarily of restricted contact type. For instance, if  $M_2$  is not exact, then  $\Sigma_1 \times M_2$  is never of restricted contact type. Nevertheless, interestingly enough, we can achieve compactness of gradient flow lines of Rabinowitz action functional for an arbitrary perturbation  $F \in \text{Ham}_c(M_1 \times M_2, \omega_1 \oplus \omega_2)$ ; accordingly Floer homology of the Rabinowitz action functional with any perturbations is well-defined.

The Künneth formula enable us to compute the Rabinowitz Floer homology of a product manifold in terms of Rabinowitz Floer homology of each manifolds. As applications, in Section 4 we shall prove the following two corollaries.

**Corollary A.** *Let  $N$  be a closed Riemannian manifold of  $\dim N \geq 2$  with  $\dim H_*(\Lambda N) = \infty$  where  $\Lambda N$  is the free loop space of  $N$ . Then there exists infinitely many leafwise intersection points for a generic  $\phi \in \text{Ham}_c(T^*S^1 \times T^*N)$  on  $(S^*S^1 \times S^*N, T^*S^1 \times T^*N)$ .*

**Remark 1.7.** Since  $(S^*S^1 \times S^*N, T^*S^1 \times T^*N)$  is of restricted contact type in the sense of Bolle (Lemma 4.6),  $\phi$  in Corollary A is not necessarily of product type. If  $\phi$  has product type, then the above result is obvious by [1, 2]. Unlike Corollary A, the following Corollary B does not assume the contact condition since Theorem B does not need any contact conditions.

**Corollary B.** *Let  $M$  be a closed and symplectically aspherical symplectic manifold and  $N$  be as above. Then a generic  $\phi \in \text{Ham}_c(T^*N \times M)$  has infinitely many leafwise intersection points on  $(S^*N \times M, T^*N \times M)$ .*

**Remark 1.8.** If  $\pi_1(N)$  is finite then  $\dim H_*(\Lambda N) = \infty$  by [30]. If the number of conjugacy classes of  $\pi_1(N)$  is infinite then  $\dim H_0(\Lambda N) = \infty$ . Therefore, the only remaining case is if  $\pi_1(N)$  is infinite but the number of conjugacy classes of  $\pi_1(N)$  is finite.

**1.1. Leafwise intersections.** Let  $(M, \omega)$  be a  $2n$  dimensional symplectic manifold and  $\Sigma$  be a coisotropic submanifold of codimension  $0 \leq k \leq n$ . Then the symplectic structure  $\omega$  determines a symplectic orthogonal bundle  $T\Sigma^\omega \subset T\Sigma$  as follows:

$$T\Sigma^\omega := \{(x, \xi) \in T\Sigma \mid \omega_x(\xi, \zeta) = 0 \text{ for all } \zeta \in T_x\Sigma\}$$

Since  $\omega$  is closed,  $T\Sigma^\omega$  is integrable, thus  $\Sigma$  is foliated by the leaves of the characteristic foliation and we denote by  $L_x$  the isotropic leaf through  $x$ . We call  $x \in \Sigma$  a *leafwise intersection point* of  $\phi \in \text{Ham}(M, \omega)$  if  $x \in L_x \cap \phi(L_x)$ . In the extremal case  $k = n$ , Lagrangian submanifold consists of only one leaf. Thus a leafwise intersection point is nothing but a Lagrangian intersection point in the case  $k = n$ . In the other extremal case that  $k = 0$ , a leafwise intersection corresponds to a periodic orbit of  $\phi$ .

The leafwise intersection problem was initiated by Moser [27] and pursued further in [9, 22, 17, 20, 16, 21, 1, 2, 3, 4, 31, 23, 24, 25, 6, 7]. We refer to [1, 24] for the history of the problem. In particular, Albers-Frauenfelder approached the problem by means of the perturbed Rabinowitz action functional and much relevant research has been conducted in [1, 2, 3, 4, 13, 14, 15, 8, 23, 24, 25, 7]. We refer to [5] for a brief survey on Rabinowitz Floer theory.

## 2. RABINOWITZ ACTION FUNCTIONAL ON PRODUCT MANIFOLDS

Since  $\Sigma_1$  and  $\Sigma_2$  are contact hypersurfaces, there exist associated Liouville vector fields  $Y_1$  resp.  $Y_2$  on  $M_1$  resp.  $M_2$  such that  $\mathcal{L}_{Y_i}\omega_i = \omega_i$  and  $Y_i \lrcorner \Sigma_i$  for  $i = 1, 2$ . We denote by  $\phi_{Y_i}^t$  the flow of  $Y_i$  and fix  $\delta > 0$  such that  $\phi_{Y_i}^t|_{\Sigma_i}$  is defined for  $|t| < \delta$ . Since  $\Sigma_1$  resp.  $\Sigma_2$  bounds a

compact region in  $M_1$  resp.  $M_2$ , we are able to define Hamiltonian functions  $G_1 \in C^\infty(M_1)$  and  $G_2 \in C^\infty(M_2)$  so that

- (1)  $G_1^{-1}(0) = \Sigma_1$  and  $G_2^{-1}(0) = \Sigma_2$  are regular level sets;
- (2)  $dG_1$  and  $dG_2$  have compact supports;
- (3)  $G_i(\phi_{Y_i}^t(x_i)) = t$  for all  $x_i \in \Sigma_i$ ,  $i = 1, 2$ , and  $|t| < \delta$ ;

We extend  $G_1, G_2$  to be defined on the whole of  $M_1 \times M_2$ :

$$\begin{aligned} \tilde{G}_i : M_1 \times M_2 &\longrightarrow \mathbb{R} & i = 1, 2 \\ (x_1, x_2) &\longmapsto G_i(x_i). \end{aligned}$$

**Definition 2.1.** Given time-dependent Hamiltonian functions  $\tilde{H}_1, \tilde{H}_2, F \in C^\infty(S^1 \times M_1 \times M_2)$ , a triple  $(\tilde{H}_1, \tilde{H}_2, F)$  is called a *Moser triple* if it satisfies

- (1) their time supports are disjoint, i.e.

$$\tilde{H}_1(t, \cdot) = \tilde{H}_2(t, \cdot) = 0 \quad \text{for } \forall t \in [0, \frac{1}{2}] \quad \text{and} \quad F(t, \cdot) = 0 \quad \text{for } \forall t \in [\frac{1}{2}, 1].$$

- (2)  $F = F_1 \oplus F_2$  for some  $F_1 \in C_c^\infty(S^1 \times M_1)$ ,  $F_2 \in C_c^\infty(S^1 \times M_2)$ .
- (3)  $\tilde{H}_1$  and  $\tilde{H}_2$  are weakly time-dependent Hamiltonian functions. That is,  $\tilde{H}_1$  and  $\tilde{H}_2$  are of the form  $(\tilde{H}_1(t, x), \tilde{H}_2(t, x)) = \chi(t)(\tilde{G}_1(x), \tilde{G}_2(x))$  for  $\chi : S^1 \rightarrow S^1$  with  $\int_0^1 \chi dt = 1$  and  $\text{Supp} \chi \subset (\frac{1}{2}, 1)$ .

Next, we recall the definition of the Hofer norm.

**Definition 2.2.** Let  $F \in C_c^\infty(S^1 \times M, \mathbb{R})$  be a compactly supported time-dependent Hamiltonian function on a symplectic manifold  $(M, \omega)$ . We set

$$\|F\|_+ := \int_0^1 \max_{x \in M} F(t, x) dt \quad \|F\|_- := - \int_0^1 \min_{x \in M} F(t, x) dt = \|-F\|_+$$

and

$$\|F\| = \|F\|_+ + \|F\|_-.$$

For  $\phi \in \text{Ham}_c(M, \omega)$  the Hofer norm is

$$\|\phi\| = \inf \{ \|F\| \mid \phi = \phi_F, F \in C_c^\infty(S^1 \times M, \mathbb{R}) \}.$$

**Lemma 2.3.** For all  $\phi \in \text{Ham}_c(M, \omega)$

$$\|\phi\| = \|\phi\| := \inf \{ \|F\| \mid \phi = \phi_F, F(t, \cdot) = 0 \quad \forall t \in [\frac{1}{2}, 1] \}.$$

PROOF. To prove  $\|\phi\| \geq \|\phi\|$ , pick a smooth monotone increasing map  $r : [0, 1] \rightarrow [0, 1]$  with  $r(0) = 0$  and  $r(1/2) = 1$ . For  $F$  with  $\phi_F = \phi$  we set  $F^r(t, x) := r'(t)F(r(t), x)$ . Then a direct computation shows  $\phi_{F^r} = \phi_F$ ,  $\|F^r\| = \|F\|$ , and  $F^r(t, x) = 0$  for all  $t \in [\frac{1}{2}, 1]$ . The reverse inequality is obvious.  $\square$

We denote by  $\mathcal{L} = \mathcal{L}_{M_1 \times M_2} \subset C^\infty(S^1, M_1 \times M_2)$  the component of contractible loops in  $M_1 \times M_2$ . With a Moser triple  $(H_1, H_2, F)$ , the perturbed Rabinowitz action functional  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(v, \eta_1, \eta_2) : \mathcal{L} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$  is defined as follows:

$$\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(v, \eta_1, \eta_2) = - \int_0^1 v^* \lambda_1 \oplus \lambda_2 - \eta_1 \int_0^1 \tilde{H}_1(t, v) dt - \eta_2 \int_0^1 \tilde{H}_2(t, v) dt - \int_0^1 F(t, v) dt \quad (2.1)$$

where  $\lambda_1 \oplus \lambda_2 = \pi_1^* \lambda_1 + \pi_2^* \lambda_2$ . The real numbers  $\eta_1$  and  $\eta_2$  can be thought of as Lagrange multipliers.

Critical points  $(v, \eta_1, \eta_2) \in \text{Crit} \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  satisfy

$$\left. \begin{aligned} \partial_t v &= \eta_1 X_{\tilde{H}_1}(t, v) + \eta_2 X_{\tilde{H}_2}(t, v) + X_F(t, v), \\ \int_0^1 \tilde{H}_1(t, v) dt &= 0, \\ \int_0^1 \tilde{H}_2(t, v) dt &= 0. \end{aligned} \right\} \quad (2.2)$$

Albers-Frauenfelder [1] observed that a critical point of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  gives rise to a leafwise intersection point. (In fact, they proved the following proposition for the codimensional one case, yet their proof continues to hold in our case, see [24] also.)

**Definition 2.4.** A leafwise coisotropic intersection point  $x \in \Sigma_1 \times \Sigma_2$  is called *periodic* if the leaf  $L_x$  contains neither a closed Reeb orbit in  $\Sigma_1$  nor a closed Reeb orbit in  $\Sigma_2$ .

**Proposition 2.5.** [1] *Let  $(v, \eta_1, \eta_2) \in \text{Crit} \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$ . Then  $x = v(1/2)$  satisfies  $\phi_F(x) \in L_x$ . Thus,  $x$  is a leafwise intersection point. Moreover, the map*

$$\text{Crit} \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2} \longrightarrow \{\text{leafwise intersections}\}$$

*is injective unless there exists a periodic leafwise intersection.*

We choose a compatible almost complex structure  $J_1$  on  $M_1$  and define the metric on  $(M_1, \omega_1)$  by  $g_1(\cdot, \cdot) = \omega_1(\cdot, J_1 \cdot)$ . Analogously we also define the metric on  $(M_2, \omega_2)$ ,  $g_2(\cdot, \cdot) = \omega_2(\cdot, J_2 \cdot)$ . Then  $g = g_1 \oplus g_2$  which is the metric on  $(M_1 \times M_2, \omega_1 \oplus \omega_2)$  induces a metric  $m$  on the tangent space  $T_{(v, \eta_1, \eta_2)}(\mathcal{L} \times \mathbb{R}^2) \cong T_v \mathcal{L} \times \mathbb{R}^2$  as follows:

$$m_{(v, \eta_1, \eta_2)}((\hat{v}^1, \hat{\eta}_1^1, \hat{\eta}_2^1), (\hat{v}^2, \hat{\eta}_1^2, \hat{\eta}_2^2)) := \int_0^1 g_v(\hat{v}^1, \hat{v}^2) dt + \hat{\eta}_1^1 \hat{\eta}_1^2 + \hat{\eta}_2^1 \hat{\eta}_2^2.$$

**Definition 2.6.** A map  $w = (v, \eta_1, \eta_2) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$  which solves

$$\partial_s w(s) + \nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w(s)) = 0 \quad (2.3)$$

is called a gradient flow line of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  with respect to the metric  $m$ .

According to Floer's interpretation, the gradient flow equation (2.3) can be interpreted as maps  $v(s, t) : \mathbb{R} \times S^1 \rightarrow M_1 \times M_2$  and  $\eta_1(s), \eta_2(s) : \mathbb{R} \rightarrow \mathbb{R}$  solving

$$\left. \begin{aligned} \partial_s v + J(v) (\partial_t v - \eta_1 X_{\tilde{H}_1}(t, v) - \eta_2 X_{\tilde{H}_2}(t, v) - X_F(t, v)) &= 0, \\ \partial_s \eta_1 - \int_0^1 \tilde{H}_1(t, v) dt &= 0, \\ \partial_s \eta_2 - \int_0^1 \tilde{H}_2(t, v) dt &= 0. \end{aligned} \right\} \quad (2.4)$$

**Definition 2.7.** The energy of a map  $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$  is defined by

$$E(w) := \int_{-\infty}^{\infty} \|\partial_s w\|_m^2 ds.$$

**Lemma 2.8.** Let  $w$  be a gradient flow line of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$ . Then

$$E(w) = \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_-) - \mathcal{A}_F^{H_1, H_2}(w_+). \quad (2.5)$$

where  $w_\pm = \lim_{s \rightarrow \pm\infty} w(s)$ .

PROOF. It follows from the gradient flow equation (2.3).

$$\begin{aligned} E(w) &= \int_{-\infty}^{\infty} m(-\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w(s)), \partial_s w(s)) ds \\ &= - \int_{-\infty}^{\infty} d\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w(s))(\partial_s w(s)) ds \\ &= - \int_{-\infty}^{\infty} \frac{d}{ds} \left( \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w(s)) \right) ds \\ &= \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_-) - \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_+). \end{aligned}$$

□

**2.1. Compactness of gradient flow lines.** In order to define Rabinowitz Floer homology, we need compactness of gradient flow lines of the Rabinowitz action functional with fixed asymptotic data. More specifically we show the following theorem. In the rest of this section, our perturbation  $F \in C_c^\infty(S^1 \times M_1 \times M_2)$  is of the form  $F_1 \oplus F_2$  for some  $F_1 \in C_c^\infty(S^1 \times M_1)$  and  $F_2 \in C_c^\infty(S^1 \times M_2)$ .

**Theorem 2.9.** Let  $\{w_n\}_{n \in \mathbb{N}}$  be a sequence of gradient flow lines of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  for which there exist  $a < b$  such that

$$a \leq \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_n(s)) \leq b, \quad \text{for all } s \in \mathbb{R}.$$

Then for every reparametrization sequence  $\sigma_n \in \mathbb{R}$ , the sequence  $w_n(\cdot + \sigma_n)$  has a subsequence which converges in  $C_{\text{loc}}^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$ .

PROOF. In order to prove the theorem, we need to verify the following three ingredients.

- (1) a uniform  $L^\infty$ -bound on  $v_n$ ,
- (2) a uniform  $L^\infty$ -bound on  $\eta_{1n}, \eta_{2n}$ ,
- (3) a uniform  $L^\infty$ -bound the derivatives of  $v_n$ .

for a sequence of gradient flow lines  $\{(v_n, \eta_{1n}, \eta_{2n})\}_{n \in \mathbb{N}}$ . Once we establish (2), the others follow by standard arguments in Floer theory. At the end of this section, we prove Theorem 2.15 which proves (2) and thus completes the proof of Theorem 2.9. □

First of all, we introduce two auxiliary action functionals  $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{L}_{M_1 \times M_2} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ :

$$\begin{aligned} \mathcal{A}_1(v, \eta_1, \eta_2) &:= \int_0^1 v^* \pi_1^* \lambda_1 - \eta_1 \int_0^1 H_1(t, v) dt - \int_0^1 F(t, v) dt, \\ \mathcal{A}_2(v, \eta_1, \eta_2) &:= \int_0^1 v^* \pi_2^* \lambda_2 - \eta_2 \int_0^1 H_2(t, v) dt - \int_0^1 F(t, v) dt. \end{aligned}$$

**Lemma 2.10.** Let  $w = (v, \eta_1, \eta_2) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$  be a gradient flow line of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  with asymptotic ends  $w_- = (v_-, \eta_{1-}, \eta_{2-})$  and  $w_+ = (v_+, \eta_{1+}, \eta_{2+})$ . Then the action values of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are bounded along  $w$  in terms of the asymptotic data:

- (i)  $\mathcal{A}_1(w(s)) \leq 2|\mathcal{A}_1(w_-)| + |\mathcal{A}_1(w_+)| + 4\|F_2\|_+, \quad \forall s \in \mathbb{R};$
- (ii)  $\mathcal{A}_2(w(s)) \leq 2|\mathcal{A}_2(w_-)| + |\mathcal{A}_2(w_+)| + 4\|F_1\|_+, \quad \forall s \in \mathbb{R}.$

PROOF. We only show the first inequality, the later one is proved in a similar way. Since it holds that  $\pi_{1*}X_F = X_{F_1}$ ,  $\pi_{2*}X_F = X_{F_2}$ , and  $i_{X_{\tilde{H}_2}}\pi_1^*\omega_1 = 0$ , we compute

$$\begin{aligned}
\frac{d}{ds}\mathcal{A}_1(w(s)) &= d\mathcal{A}_1(w(s))[\partial_s w(s)] \\
&= \int_0^1 \pi_1^*\omega_1(\partial_t v, \partial_s v) - \int_0^1 \omega_1 \oplus \omega_2(\eta_1 X_{H_1}(t, v) + X_F(t, v), \partial_s v) \\
&\quad - \left( \int_0^1 \tilde{H}_1(t, v) dt \right)^2 \\
&= \int_0^1 \pi_1^*\omega_1(\partial_t v - \eta_1 X_{\tilde{H}_1}(t, v) - X_F(t, v), \partial_s v) dt \\
&\quad - \int_0^1 \pi_2^*\omega_2(X_F(t, v), \partial_s v) dt - \left( \int_0^1 \tilde{H}_1(t, v) dt \right)^2 \\
&= - \int_0^1 \pi_1^*\omega_1(\partial_s v, J\partial_s v) dt - \int_0^1 \frac{d}{ds} F_2(t, \pi_2 \circ v) dt - \left( \int_0^1 \tilde{H}_1(t, v) dt \right)^2.
\end{aligned}$$

Integrating the above equality from  $-\infty$  to any  $s_0 \in \mathbb{R}$ , we have

$$\begin{aligned}
\mathcal{A}_1(w(s_0)) - \mathcal{A}_1(w_-) &= \int_{-\infty}^{s_0} \frac{d}{ds} \mathcal{A}_1(w(s)) ds \\
&= - \int_{-\infty}^{s_0} \int_0^1 \pi_1^*\omega_1(\partial_s v, J\partial_s v) dt ds \\
&\quad - \int_{-\infty}^{s_0} \int_0^1 \frac{d}{ds} F_2(t, \pi_2 \circ v) dt ds - \int_{-\infty}^{s_0} \left( \int_0^1 \tilde{H}_1(t, v) dt \right)^2 ds \\
&= - \int_{-\infty}^{s_0} \mathbf{B}(s) ds - \int_0^1 F_2(t, \pi_2 \circ v(s_0)) - F_2(t, \pi_2 \circ v_-) dt.
\end{aligned} \tag{2.6}$$

where  $\mathbf{B}(s)$  is defined as

$$\mathbf{B}(s) := \int_0^1 \pi_1^*\omega_1(\partial_s v, J\partial_s v) dt + \left( \int_0^1 \tilde{H}_1(t, v) dt \right)^2.$$

Therefore the following estimate can be derived for any  $s_0 \in \mathbb{R}$

$$|\mathcal{A}_1(w(s_0))| \leq |\mathcal{A}_1(w_+)| + 2\|F_2\|_+ + \left| \int_{-\infty}^{s_0} \mathbf{B}(s) ds \right|,$$

and it remains to find a bound for  $|\int_{-\infty}^{s_0} \mathbf{B}(s) ds|$ . Since  $\mathbf{B}(s)$  is nonnegative, we are able to estimate as the following. By setting  $s_0 = \infty$  in formula (2.6), we have

$$\mathcal{A}_1(w_+) - \mathcal{A}_1(w_-) = - \int_{-\infty}^{\infty} \mathbf{B}(s) ds - \int_0^1 F_2(t, \pi_2 \circ v_+) - F_2(t, \pi_2 \circ v_-) dt$$

Using the above formula, we obtain

$$\begin{aligned} \left| \int_{-\infty}^{s_0} \mathbf{B}(s) ds \right| &\leq \left| \int_{-\infty}^{\infty} \mathbf{B}(s) ds \right| \\ &\leq |\mathcal{A}_1(w_+)| + |\mathcal{A}_1(w_-)| + 2\|F_2\|_+. \end{aligned}$$

Thus we finally deduce

$$|\mathcal{A}_1(w(s_0))| \leq |\mathcal{A}_1(w_+)| + 2|\mathcal{A}_1(w_-)| + 4\|F_2\|_+, \quad \forall s_0 \in \mathbb{R}.$$

□

Once we have Lemma 2.10, the rest of the proof of Theorem 2.9 is quite similar as in [1].

**Lemma 2.11.** Given a gradient flow line  $w(s) = (v, \eta_1, \eta_2)(s) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$  of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$ , assume that  $v(t) \in U_\delta := \tilde{G}_1^{-1}(-\delta, \delta) \cap \tilde{G}_2^{-1}(-\delta, \delta)$  for all  $t \in (1/2, 1)$  with  $0 < 2\delta < \min\{1, \delta_0\}$ . Then there exists  $C_i > 0$  satisfying

$$|\eta_i| \leq C_i \left( |\mathcal{A}_i(v, \eta)| + \|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}\|_m + 1 \right), \quad i = 1, 2.$$

PROOF. We estimate

$$\begin{aligned} |\mathcal{A}_i(v, \eta_1, \eta_2)| &= \left| \int_0^1 v^* \pi_i^* \lambda_i + \eta_i \int_0^1 \tilde{H}_i(t, v) dt + \int_0^1 F(t, v) dt \right| \\ &\geq \left| \eta_i \int_0^1 \pi_i^* \lambda_i(v)(X_{\tilde{H}_i}(t, v)) dt \right| - \left| \int_0^1 \pi_i^* \lambda_i(v)(X_F(t, v)) dt \right| - \left| \eta_i \int_{\frac{1}{2}}^1 \tilde{H}_i(t, v) dt \right| \\ &\quad - \left| \int_0^{\frac{1}{2}} F(t, v) dt \right| - \left| \int_0^1 \pi_i^* \lambda_i(v)(\partial_t v - \eta_1 X_{\tilde{H}_1}(t, v) - \eta_2 X_{\tilde{H}_2}(t, v) - X_F(t, v)) dt \right| \\ &\geq |\eta_i| - \delta |\eta_i| - C_{i, \delta} \|\partial_t v - \eta_1 X_{\tilde{H}_1}(t, v) - \eta_2 X_{\tilde{H}_2}(t, v) - X_F(t, v)\|_{L^1} - C_{i, \delta, F} \\ &\geq |\eta_i| - \delta |\eta_i| - C_{i, \delta} \|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}\|_m - C_{i, F} \end{aligned}$$

where  $C_{i, \delta} := \|\pi_i^* \lambda_i|_{U_\delta}\|_{L^\infty}$  and  $C_{i, \delta, F} := \|F\|_{L^\infty} + C_i \|X_F\|_{L^\infty}$ . The second inequality holds since  $\pi_i^* \lambda_i(X_{\tilde{H}_j}) = 0$  if  $i \neq j$ . This estimate finishes the lemma with

$$C_i := \max \left\{ \frac{1}{1 - \delta}, \frac{C_{i, \delta}}{1 - \delta}, \frac{C_{i, \delta, F}}{1 - \delta} \right\}, \quad i = 1, 2.$$

□

**Lemma 2.12.** Given a gradient flow line  $w(s) = (v, \eta_1, \eta_2)(s) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$  of  $\mathcal{A}_F^{H_1, H_2}$ , if there exists  $t \in (\frac{1}{2}, 1)$  such that  $v(t) \notin U_\delta$  then  $\|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(v, \eta_1, \eta_2)\|_m > \epsilon$  for some  $\epsilon = \epsilon_\delta$ .

PROOF. Since  $v(t) \notin U_\delta$  for some  $t \in (\frac{1}{2}, 1)$ , either  $v(t) \notin U_\delta^1 := \tilde{G}_1^{-1}(-\delta, \delta)$  or  $v(t) \notin U_\delta^2 := \tilde{G}_2^{-1}(-\delta, \delta)$  for that  $t \in (\frac{1}{2}, 1)$ . For simplicity, suppose  $v(t) \notin U_\delta^1$ . If in addition  $v(t) \notin U_{\delta/2}^1$  for all  $t \in (\frac{1}{2}, 1)$ , then we easily conclude that

$$\|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(v, \eta_1, \eta_2)\|_m \geq \left| \int_0^1 \tilde{H}_1(t, v(t)) dt \right| = \left| \int_{\frac{1}{2}}^1 \tilde{H}_1(t, v(t)) dt \right| \geq \frac{\delta}{2}.$$

Otherwise there is  $t' \in (\frac{1}{2}, 1)$  such that  $v(t') \in U_{\delta/2}^1$ . Thus there exist  $t_0, t_1 \in (\frac{1}{2}, 1)$  satisfying one of the following two cases.

$$v(t_0) \in \partial U_{\delta/2}^1, \quad v(t_1) \in \partial U_\delta^1 \quad \text{and} \quad v(s) \in U_\delta^1 - U_{\delta/2}^1 \quad \text{for all } s \in [t_0, t_1] \quad (2.7)$$



or

$$v(t_1) \in \partial U_\delta^1, \quad v(t_0) \in \partial U_{\delta/2}^1 \quad \text{and} \quad v(s) \in U_\delta^1 - U_{\delta/2}^1 \quad \text{for all } s \in [t_1, t_0].$$

We only treat the first case (2.7) and the second case follows analogously. With

$$\kappa := \max_{x \in U_\delta} \|\nabla_g \tilde{G}_1(x)\|_g$$

we estimate

$$\begin{aligned} & \kappa \|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(v, \eta_1, \eta_2)\|_m \\ & \geq \kappa \|\partial_t v - \eta_1 X_{\tilde{H}_1}(t, v) - \eta_2 X_{\tilde{H}_2}(t, v) - X_F(t, v)\|_{L^2} \\ & \geq \kappa \|\partial_t v - \eta_1 X_{\tilde{H}_1}(t, v) - \eta_2 X_{\tilde{H}_2}(t, v) - X_F(t, v)\|_{L^1} \\ & \geq \int_{t_0}^{t_1} \|\partial_t v - \eta_1 X_{\tilde{H}_1}(t, v) - \eta_2 X_{\tilde{H}_2}(t, v) - X_F(t, v)\|_g \cdot \|\nabla \tilde{G}_1(x)\|_g dt \\ & \geq \left| \int_{t_0}^{t_1} \langle \nabla \tilde{G}_1(v(t)), \partial_t v - \eta_1 X_{\tilde{H}_1}(t, v) - \eta_2 X_{\tilde{H}_2}(t, v) - X_F(t, v) \rangle dt \right| \\ & = \left| \int_{t_0}^{t_1} d\tilde{G}_1(v(t)) (\partial_t v - \eta_1 X_{\tilde{H}_1}(t, v) - \eta_2 X_{\tilde{H}_2}(t, v) - \underbrace{X_F(t, v)}_{=0}) dt \right| \\ & = \left| \int_{t_0}^{t_1} \frac{d}{dt} \tilde{G}_1(v(t)) dt - \underbrace{d\tilde{G}_1(v(t)) (\eta_1 X_{\tilde{H}_1}(t, v) - \eta_2 X_{\tilde{H}_2}(t, v))}_{=\eta_1 \chi \omega(X_{\tilde{G}_1}, X_{\tilde{G}_1}) + \eta_2 \chi \omega(X_{\tilde{G}_1}, X_{\tilde{G}_2}) = 0} dt \right| \\ & \geq |\tilde{G}_1(v(t_1))| - |\tilde{G}_1(v(t_0))| \\ & = \frac{\delta}{2}. \end{aligned}$$

Hence, the lemma follows with  $\epsilon_\delta := \min\{\delta/2, \delta/(2\kappa)\}$ . □

Combining Lemma 2.11 and Lemma 2.12, we deduce the following *fundamental lemma*.

**Lemma 2.13.** For a gradient flow line  $w(s) = (v, \eta_1, \eta_2)(s) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$  of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$ , the following assertion holds for  $i = 1, 2$  with some  $C, \epsilon > 0$ .

$$|\eta_i| \leq C(|\mathcal{A}_i(w_-)| + |\mathcal{A}_i(w_+)| + 1) \quad \text{if} \quad \|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(v, \eta_1, \eta_2)\|_m < \epsilon.$$

PROOF. According to Lemma 2.12,  $v(t)$  lies in  $U_\delta$  for all  $t \in (\frac{1}{2}, 1)$  under the assumption that  $\|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(v, \eta_1, \eta_2)\|_m < \epsilon$ . Thus we are able to apply Lemma 2.11 and the following computation concludes the proof of the lemma.

$$\begin{aligned} |\eta_i| & \leq C_i(|\mathcal{A}_i(v, \eta_1, \eta_2)| + \|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(v, \eta_1, \eta_2)\|_m + 1) \\ & \leq C_i(2|\mathcal{A}_i(w_-)| + |\mathcal{A}_i(w_+)| + 4\|F_2\|_+ + \|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(v, \eta_1, \eta_2)\|_m + 1) \\ & \leq C_i(2|\mathcal{A}_i(w_-)| + |\mathcal{A}_i(w_+)| + 4\|F_2\|_+ + 1 + \epsilon). \end{aligned}$$

□

**Lemma 2.14.** For given a gradient flow line  $w$  of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  and  $\sigma \in \mathbb{R}$ , we define

$$\tau(\sigma) := \inf\{\tau \geq 0 \mid \|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w(\sigma + \tau))\|_m \leq \epsilon\},$$

Then we obtain a bound on  $\tau(\sigma)$  as follows:

$$\tau(\sigma) \leq \frac{\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_-) - \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_+)}{\epsilon^2}.$$

PROOF. Using Lemma 2.8, we compute

$$\begin{aligned} \epsilon^2 \tau(\sigma) &\leq \int_{\sigma}^{\sigma + \tau(\sigma)} \|\nabla_m \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w)\|_m^2 ds \\ &\leq E(w) \\ &\leq \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_-) - \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_+). \end{aligned}$$

Dividing both sides through by  $\epsilon^2$ , the lemma follows.  $\square$

**Theorem 2.15.** *Given two critical points  $w_-$  and  $w_+$ , there exists a constant  $\Theta > 0$  depending only on  $w_-$  and  $w_+$  such that every gradient flow line  $w(s) = (v, \eta_1, \eta_2)(s)$  of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  with fixed asymptotic ends  $w_{\pm}$  satisfies*

$$\|\eta_i\|_{L^\infty} \leq \Theta \quad \text{for } i = 1, 2.$$

PROOF. Using Lemma 2.10 and Lemma 2.14, we estimate

$$\begin{aligned} |\eta_i(\sigma)| &\leq |\eta_i(\sigma + \tau(\sigma))| + \int_{\sigma}^{\sigma + \tau(\sigma)} |\partial_s \eta_i(s)| ds \\ &\leq C(|\mathcal{A}_i(w_-)| + |\mathcal{A}_i(w_+)| + 1) + \tau(\sigma) \|\tilde{H}_i\|_{L^\infty} \\ &\leq C(|\mathcal{A}_i(w_-)| + |\mathcal{A}_i(w_+)| + 1) + \left( \frac{\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_-) - \mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}(w_+)}{\epsilon^2} \right) \|H_i\|_{L^\infty}. \end{aligned}$$

$\square$

As we mentioned before, Theorem 2.15 completes the proof of Theorem 2.9.

### 3. KÜNNETH FORMULA IN RABINOWITZ FLOER HOMOLOGY

Thanks to the previous section, we are now able to define Rabinowitz Floer homology of  $(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$  for admissible perturbations of the form  $F_1 \oplus F_2$  (or unperturbed). Whilst  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  is generically Morse (Lemma 4.7),  $\mathcal{A}^{H_1, H_2}$  is never Morse because there is a  $S^1$ -symmetry coming from time-shift on the critical point set. However  $\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}$  is generically Morse-Bott, so we are able to compute its Floer homology by choosing an auxiliary Morse function on the critical manifold and counting gradient flow lines with cascades, see [18, 13]. Using the continuation method in Floer theory, we know that the Floer homology of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  is isomorphic to the Floer homology of  $\mathcal{A}^{\tilde{H}_1, \tilde{H}_2} = \mathcal{A}_0^{\tilde{H}_1, \tilde{H}_2}$  whenever these Floer homologies are defined. Thus we only treat the unperturbed Rabinowitz action functional  $\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}$  and its Floer homology. Furthermore, we derive the Künneth formula by making use of the fact that all critical points and gradient flow lines can be split. In the last subsection, we prove Theorem B using similar steps to those in the proof of Theorem A; but we need to prove a

special version of an isoperimetric inequality (Lemma 3.2) since unlike Theorem A, we have not insisted on any restrictions on perturbations in Theorem B.

**3.1. Rabinowitz Floer homology.** Firstly, we define a chain complex and a boundary operator for the Rabinowitz action functional. In order to define a chain complex we choose an additional Morse function  $f$  on the critical manifold  $\text{Crit}\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}$ . We define a  $\mathbb{Z}/2$ -Floer chain complex

$$\text{CF}_n(\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}, f) := \left\{ \xi = \sum_{(v, \eta_1, \eta_2)} \xi_{(v, \eta_1, \eta_2)}(v, \eta_1, \eta_2) \mid (v, \eta_1, \eta_2) \in \text{Crit}_n f, \xi_{(v, \eta_1, \eta_2)} \in \mathbb{Z}/2 \right\}$$

where  $\xi_{(v, \eta_1, \eta_2)}$  satisfy the finiteness condition:

$$\#\{(v, \eta_1, \eta_2) \in \text{Crit}_n f \mid \xi_{(v, \eta_1, \eta_2)} \neq 0, \mathcal{A}^{H_1, H_2}(v, \eta_1, \eta_2) \geq \kappa\} < \infty, \quad \forall \kappa \in \mathbb{R}.$$

The grading for the chain complex,  $\mu = \mu_{\text{RFH}}$ , is described in the appendix of this paper.

To define the boundary operator, we roughly explain the notion of a *gradient flow line with cascades*. For rigorous and explicit constructions, we refer to [18]. Consider a gradient flow line with cascades interchanging  $w_- \subset C^-$  and  $w_+ \subset C^+$  where  $w_{\pm} \in \text{Crit} f$  and  $C^{\pm} \subset \text{Crit}\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}$ ; it starts with a gradient flow line of  $f$  in  $C^-$  with the negative asymptotic end  $w_-$  and meets the negative asymptotic ends of a gradient flow line of  $\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}$  (solving (2.4) with  $F \equiv 0$ ). We refer to this gradient flow line as a *cascade*. Its positive asymptotic end encounters a gradient flow line of  $f$  in  $C^+$  which converges to  $w_+$ . Several cascades and no cascades are also allowed. Now, we define a moduli space

$$\widehat{\mathcal{M}}\{w_-, w_+\} := \left\{ w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2) \mid \begin{array}{l} w \text{ is a gradient flow line with cascades} \\ \text{with } \lim_{s \rightarrow \pm\infty} w(s) = w_{\pm} \in \text{Crit} f \end{array} \right\}$$

and divide out the  $\mathbb{R}$ -action from shifting the gradient flow lines in the  $s$ -variable. Then we obtain the moduli space of unparametrized gradient flow lines, denoted by

$$\mathcal{M} := \widehat{\mathcal{M}}/\mathbb{R}.$$

The standard transversality theory shows that this moduli space is a smooth manifold for a generic choice of the almost complex structure and the metric, see [19, 12]. From the calculation (5.1) in the appendix, we also know that the dimension of  $\mathcal{M}$  is equal to  $\mu_{\text{RFH}}(w_-) - \mu_{\text{RFH}}(w_+) - 1$ . Therefore if  $\mu_{\text{RFH}}(w_-) - \mu_{\text{RFH}}(w_+) = 1$ ,  $\mathcal{M}(w_-, w_+)$  is a finite set because of Theorem 2.9. We let  $\#_2 \mathcal{M}\{w_-, w_+\}$  be the parity of this moduli space. We define the boundary maps  $\{\partial_n^{\tilde{H}_1, \tilde{H}_2}\}_{n \in \mathbb{Z}}$  as follows:

$$\begin{aligned} \partial_{n+1}^{\tilde{H}_1, \tilde{H}_2} : \text{CF}_{n+1}(\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}) &\longrightarrow \text{CF}_n(\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}) \\ w_- &\longmapsto \sum_{w_+ \in \text{Crit}_n f} \#_2 \mathcal{M}\{w_-, w_+\} w_+. \end{aligned}$$

Due to the Floer's central theorem, we know that  $\partial_n^{\tilde{H}_1, \tilde{H}_2} \circ \partial_{n+1}^{H_1, H_2} = 0$  so that  $(\text{CF}_*(\mathcal{A}^{H_1, H_2}), \partial_*^{H_1, H_2})$  is a chain complex. We define Rabinowitz Floer homology by

$$\text{RFH}_n(\Sigma_1 \times \Sigma_2, M_1 \times M_2) := \text{HF}_n(\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}) = \text{H}_n(\text{CF}_*(\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}), \partial_*^{\tilde{H}_1, \tilde{H}_2}).$$

**Remark 3.1.** Since in the previous section, we achieved the compactness result for  $\mathcal{A}_{F_1 \oplus F_2}^{\tilde{H}_1, \tilde{H}_2}$ , the Floer homology  $\mathrm{HF}_n(\mathcal{A}_{F_1 \oplus F_2}^{\tilde{H}_1, \tilde{H}_2})$  can be defined; besides, it is isomorphic to  $\mathrm{RFH}_n(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$  by the continuation homomorphism which counts gradient flow lines of  $\mathcal{A}_{F_s}^{\tilde{H}_1, \tilde{H}_2}$  where  $F_s$  is a homotopy between  $F_1 \oplus F_2$  and  $F \equiv 0$ .

**3.2. Proof of Theorem A.** At first, we set

$$H_1(t, x_1) = \chi(t)G_1(x_1) \in C^\infty(S^1 \times M_1), \quad H_2(t, x_2) = \chi(t)G_2(x_2) \in C^\infty(S^1 \times M_2)$$

where  $\chi : S^1 \rightarrow [0, \infty)$  with  $\int_0^1 \chi dt = 1$  and  $\mathrm{Supp} \chi \subset (\frac{1}{2}, 1)$ ; it is clear that

$$(\pi_i)_* X_{\tilde{H}_i}(x_1, x_2) = X_{H_i}(x_i), \quad i = 1, 2.$$

We consider the Rabinowitz action functionals  $\mathcal{A}^{H_1} : \mathcal{L}_{M_1} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\mathcal{A}^{H_2} : \mathcal{L}_{M_2} \times \mathbb{R} \rightarrow \mathbb{R}$ :

$$\begin{aligned} \bullet \quad \mathcal{A}^{H_1}(v_1, \eta_1) &= - \int_0^1 v_1^* \lambda_1 - \eta_1 \int_0^1 H_1(t, v_1) dt, \\ \bullet \quad \mathcal{A}^{H_2}(v_2, \eta_2) &= - \int_0^1 v_2^* \lambda_2 - \eta_2 \int_0^1 H_2(t, v_2) dt. \end{aligned}$$

In fact, we can accomplish compactness of gradient flow lines of each action functional with minor modifications of our case, or see [1]. We observe that  $(v_1, \eta_1) \in \mathrm{Crit} \mathcal{A}^{H_1}$  solves

$$\partial_t v_1 = \eta_1 X_{H_1}(t, v_1) \quad \& \quad \int_0^1 H_1(t, v_1) dt = 0, \quad (3.1)$$

and  $(v_2, \eta_2) \in \mathrm{Crit} \mathcal{A}^{H_2}$  solves

$$\partial_t v_2 = \eta_2 X_{H_2}(t, v_2) \quad \& \quad \int_0^1 H_2(t, v_2) dt = 0. \quad (3.2)$$

Moreover a gradient flow line  $w_1(s, t) = (v_1(s, t), \eta_1(s)) : \mathbb{R} \times S^1 \rightarrow M_1 \times \mathbb{R}$  resp.  $w_2(s, t) = (v_2(s, t), \eta_2(s)) : \mathbb{R} \times S^1 \rightarrow M_2 \times \mathbb{R}$  is characterized by

$$\begin{aligned} \bullet \quad \partial_s v_1 + J_1(v_1)(\partial_t v_1 - \eta_1 X_{H_1}(t, v_1)) &= 0, \quad \partial_s \eta_1 - \int_0^1 H_1(t, v_1) dt = 0, \quad \text{resp.} \\ \bullet \quad \partial_s v_2 + J_2(v_2)(\partial_t v_2 - \eta_2 X_{H_2}(t, v_2)) &= 0, \quad \partial_s \eta_2 - \int_0^1 H_2(t, v_2) dt = 0. \end{aligned} \quad (3.3)$$

Then we define chain complexes  $\mathrm{CF}(\mathcal{A}^{H_1})$ ,  $\mathrm{CF}(\mathcal{A}^{H_2})$  and their boundary operators  $\partial^{H_1}$ ,  $\partial^{H_2}$  analogously as before, or see [1] and denote their Floer homologies by

$$\mathrm{RFH}(\Sigma_1, M_1) = \mathrm{H}(\mathrm{CF}(\mathcal{A}^{H_1}), \partial^{H_1}), \quad \mathrm{RFH}(\Sigma_2, M_2) = \mathrm{H}(\mathrm{CF}(\mathcal{A}^{H_2}), \partial^{H_2}).$$

Next, for the Künneth formula, we define the tensor product of chain complexes by

$$(\mathrm{CF}_*(\mathcal{A}^{H_1}) \otimes \mathrm{CF}_*(\mathcal{A}^{H_2}))_n = \bigoplus_{i=0}^n \mathrm{CF}_i(\mathcal{A}^{H_1}) \otimes \mathrm{CF}_{n-i}(\mathcal{A}^{H_2}).$$

together with the boundary operator  $\partial_n^\otimes$  given by

$$\partial_n^\otimes((v_1, \eta_1)_i \otimes (v_2, \eta_2)_{n-i}) = \partial_i^{H_1}(v_1, \eta_1)_i \otimes (v_2, \eta_2)_{n-i} + (v_1, \eta_1)_i \otimes \partial_{n-i}^{H_2}(v_2, \eta_2)_{n-i}.$$

Analyzing the critical point equations (2.2) when  $F \equiv 0$ , (3.1), and (3.2), we easily notice that  $((v_1, v_2), \eta_1, \eta_2) = (v, \eta_1, \eta_2) \in \mathrm{Crit} \mathcal{A}^{H_1, H_2}$  if and only if  $(v_1, \eta_1) \in \mathrm{Crit} \mathcal{A}^{H_1}$  and  $(v_2, \eta_2) \in$

$\text{Crit}\mathcal{A}^{H_2}$  where  $v_1 = \pi_1 \circ v : S^1 \rightarrow M_1$  and  $v_2 = \pi_2 \circ v : S^1 \rightarrow M_2$  for the projections  $\pi_1, \pi_2$ . Here,  $(v_1, v_2) \in C^\infty(S^1, M_1 \times M_2)$  is defined by

$$\begin{aligned} (v_1, v_2) : S^1 &\longrightarrow M_1 \times M_2, \\ t &\longmapsto (v_1(t), v_2(t)). \end{aligned}$$

Moreover the index behaves additively (see (5.2)), thus we have

$$\text{Crit}_n(\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}) = \bigcup_{i+j=n} \text{Crit}_i(\mathcal{A}^{H_1}) \times \text{Crit}_j(\mathcal{A}^{H_2}),$$

and we are able to define a chain homomorphism:

$$\begin{aligned} P_n : (\text{CF}_*(\mathcal{A}^{H_1}) \otimes \text{CF}_*(\mathcal{A}^{H_2}))_n &\longrightarrow \text{CF}_n(\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}), \\ (v_1, \eta_1) \otimes (v_2, \eta_2) &\longmapsto ((v_1, v_2), \eta_1, \eta_2). \end{aligned}$$

To verify that  $P_n$  is a chain homomorphism, we need to show that

$$\partial_n^{H_1, H_2} \circ P_n = P_{n-1} \circ \partial_n^\otimes.$$

For  $w_{1-} = (v_{1-}, \eta_{1-}) \in \text{Crit}\mathcal{A}^{H_1}$  and  $w_{2-} = (v_{2-}, \eta_{2-}) \in \text{Crit}\mathcal{A}^{H_2}$ , we compute

$$\begin{aligned} \partial_n^{\tilde{H}_1, \tilde{H}_2} \circ P_n(w_{1-} \otimes w_{2-}) &= \partial_n^{\tilde{H}_1, \tilde{H}_2} \underbrace{((v_{1-}, v_{2-}), \eta_{1-}, \eta_{2-})}_{=: w_-} \\ &= \sum_{\substack{w_+ \in \text{Crit}\mathcal{A}^{\tilde{H}_1, \tilde{H}_2}; \\ \mu(w_+) = \mu(w_-) - 1}} \#_2 \mathcal{M}\{w_-, w_+\} w_+ \\ &= \sum_{\substack{(v_{1+}, \eta_{1+}) \in \text{Crit}\mathcal{A}^{H_1}; \\ \mu(w_{1+}) = \mu(w_{1-}) - 1}} \#_2 \mathcal{M}\{w_-, ((v_{1+}, v_{2-}), \eta_{1+}, \eta_{2-})\} ((v_{1+}, v_{2-}), \eta_{1+}, \eta_{2-}) \\ &\quad + \sum_{\substack{(v_{2+}, \eta_{2+}) \in \text{Crit}\mathcal{A}^{H_2}; \\ \mu(w_{2+}) = \mu(w_{2-}) - 1}} \#_2 \mathcal{M}\{w_-, ((v_{1-}, v_{2+}), \eta_{1-}, \eta_{2+})\} ((v_{1-}, v_{2+}), \eta_{1-}, \eta_{2+}) \\ &= \sum_{\substack{(v_{1+}, \eta_{1+}) \in \text{Crit}\mathcal{A}^{H_1}; \\ \mu(w_{1+}) = \mu(w_{1-}) - 1}} \#_2 \mathcal{M}\{w_{1-}, w_{1+}\} P_{n-1}(w_{1+} \otimes w_{2-}) \\ &\quad + \sum_{\substack{(v_{2+}, \eta_{2+}) \in \text{Crit}\mathcal{A}^{H_2}; \\ \mu(w_{2+}) = \mu(w_{2-}) - 1}} \#_2 \mathcal{M}\{w_{2-}, w_{2+}\} P_{n-1}(w_{1-} \otimes w_{2+}) \\ &= P_{n-1}(\partial_i^{H_1} w_{1-} \otimes w_{2-}) + P_{n-1}(w_{1-} \otimes \partial_{n-i}^{H_2} w_{2-}) \\ &= P_{n-1} \circ \partial_n^\otimes(w_{1-} \otimes w_{2-}). \end{aligned}$$

where  $\mathcal{M}\{w_{1-}, w_{1+}\}$  resp.  $\mathcal{M}\{w_{2-}, w_{2+}\}$  is the moduli space which consists of gradient flow lines with cascades of  $\mathcal{A}^{H_1}$  resp.  $\mathcal{A}^{H_2}$ . The fourth equality follows by comparing (2.4) together with (3.3). Therefore we have an isomorphism

$$(P_\bullet)_* : \text{H}_\bullet(\text{CF}(\mathcal{A}^{H_1}) \otimes \text{CF}(\mathcal{A}^{H_2})) \xrightarrow{\cong} \text{H}_\bullet(\text{CF}(\mathcal{A}^{\tilde{H}_1, \tilde{H}_2})) = \text{RFH}_\bullet(\Sigma_1 \times \Sigma_2, M_1 \times M_2).$$

Finally, the algebraic Künneth formula enable us to derive the desired (topological) Künneth formula in Rabinowitz Floer homology.

$$\mathrm{RFH}_n(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \cong \bigoplus_{p=0}^n \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{RFH}_{n-p}(\Sigma_2, M_2).$$

**3.3. Proof of Theorem B.** In this subsection, we do not consider  $\Sigma_2$  and let  $(M_2, \omega_2)$  be closed and symplectically aspherical, i.e.  $\omega_2|_{\pi_2(M_2)}$ . To prove Statement (B1) in Theorem B, we need compactness of gradient flow lines of the perturbed Rabinowitz action functional on  $(\Sigma_1 \times M_2, M_1 \times M_2)$  for an arbitrary perturbation  $F \in C_c^\infty(S^1 \times M_1 \times M_2)$ . For that reason, we analyze the Rabinowitz action functional again as in Section 4; once we obtain the fundamental lemma, then the remaining steps are exactly same as before. Moreover due to compactness of gradient flow lines, we can find a leafwise intersection point for Hofer-small Hamiltonian diffeomorphisms using the stretching the neck argument in [1]. We assume that  $\Sigma_1 \times M_2$  bounds a compact region in  $M_1 \times M_2$  for Statement (B2) in Theorem B throughout this subsection; but, when it comes to the existence of leafwise intersections,  $\Sigma_1 \times M_2$  need not bound a compact region in  $M_1 \times M_2$  using the techniques in [23, 24]. As before, we choose a defining Hamiltonian function  $G \in C^\infty(M_1)$  so that

- (1)  $G^{-1}(0) = \Sigma_1$  is a regular level set and  $dG$  has a compact support.
- (2)  $G_i(\phi_Y^t(x)) = t$  for all  $x \in \Sigma_i$ , and  $|t| < \delta$ ;

where  $Y$  is the Liouville vector field for  $\Sigma_1 \subset M_1$ . We define  $\tilde{G} \in C^\infty(M_1 \times M_2)$  by  $\tilde{G}(x_1, x_2) = G(x_1)$ . Thus  $\tilde{G}$  is a defining Hamiltonian function for  $\Sigma_1 \times M_2$ . We let  $\tilde{H}(t, x) = \chi(t)\tilde{G}(x) \in C^\infty(S^1 \times M_1 \times M_2)$ . With a perturbation  $F \in C_c^\infty(S^1 \times M_1 \times M_2)$ , the perturbed Rabinowitz action functional  $\mathcal{A}_F^{\tilde{H}} : \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R}$  is given by

$$\mathcal{A}_F^{\tilde{H}}(v, \eta) = - \int_{D^2} \bar{v}^* \omega_1 \oplus \omega_2 - \eta \int_0^1 \tilde{H}(t, v) dt - \int_0^1 F(t, v) dt$$

where  $\mathcal{L} = \mathcal{L}_{M_1 \times M_2} \subset C^\infty(S^1, M_1 \times M_2)$  is the component of contractible loops in  $M_1 \times M_2$  and  $\bar{v} : D^2 \rightarrow M_1 \times M_2$  is a filling disk of  $v$ . The symplectic asphericity condition implies that the value of the above action functional is independent of the choice of filling disc.

Next, we prove the following lemma using a kind of isoperimetric inequality.

**Lemma 3.2.** Let  $w(s, t) = (v(s, t), \eta(s)) \in C^\infty(\mathbb{R} \times S^1, M_1 \times M_2) \times C^\infty(\mathbb{R}, \mathbb{R})$  be a gradient flow line of  $\mathcal{A}_F^{\tilde{H}}$ . We set  $\gamma(t) = v(s_0, t) \in C^\infty(S^1, M_1 \times M_2)$  for some fixed  $s_0 \in \mathbb{R}$ . Then  $\int_{D^2} \bar{\gamma}^* \pi_2^* \omega_2$  is uniformly bounded provided  $\|\nabla_m \mathcal{A}_F^{\tilde{H}}(v(s_0, \cdot), \eta(s_0))\|_m < \epsilon$  for some  $\epsilon > 0$ :

$$\left| \int_{D^2} \bar{\gamma}^* \pi_2^* \omega_2 \right| \leq \max_{x \in \widetilde{M}_2} \{ \|\lambda_{\widetilde{M}_2}(x)\|_{\tilde{g}_2} \mid d_{\tilde{g}_2}(x, \widetilde{M}_\star) < \epsilon + \|X_F\|_{L^\infty} \} (\epsilon + \|X_F\|_{L^\infty}). \quad (3.4)$$

where  $\widetilde{M}_2$  is the universal covering of  $M_2$ ;  $\tilde{g}_2$  is the lifting of the metric  $g_2(\cdot, \cdot) = \omega_2(\cdot, J_2 \cdot)$  on  $M_2$ ;  $\widetilde{M}_\star$  is a fundamental domain in  $\widetilde{M}_2$ ;  $d_{\tilde{g}_2}(x, \widetilde{M}_\star)$  is the distance between  $x$  and  $\widetilde{M}_\star$ ; the value on the right hand side of (3.4) is finite since  $\widetilde{M}_\star \cong M_2$  is compact.

**PROOF.** We write  $v(s, t)$  as  $v(s, t) = (v_1, v_2)(s, t)$  where  $v_1 : \mathbb{R} \times S^1 \rightarrow M_1$  and  $v_2 : \mathbb{R} \times S^1 \rightarrow M_2$ . Let  $\gamma \in C^\infty(S^1, M_1 \times M_2)$  be defined by  $\gamma(t) = v(s_0, t)$  for some  $s_0 \in \mathbb{R}$ . Since  $\gamma$  is contractible and  $M_2$  is symplectically aspherical, the value of  $\int_{D^2} \bar{\gamma}^* \pi_2^* \omega_2$  is well-defined. Let  $\gamma_2 := \pi_2 \circ \gamma$ . We also consider  $(\widetilde{M}_2, \widetilde{\omega}_2)$  the universal cover of  $M_2$  where  $\widetilde{\omega}_2$  is the lift

of  $\omega_2$  and we also lift the metric  $g_2$  on  $M_2$  which we write as  $\tilde{g}_2$ . Since we have assumed the symplectically asphericity of  $(M_2, \omega_2)$ , there exists a primitive one form  $\lambda_{\tilde{M}_2}$  of  $\tilde{\omega}_2$ . Let  $\tilde{M}_\star (\cong M_2)$  be one of the fundamental domains in  $\tilde{M}_2$  and  $\tilde{v}(s, t) : \mathbb{R} \times S^1 \rightarrow M_1 \times \tilde{M}_2$  be the lift of  $v$  such that  $\tilde{v}(s_0, t) = \tilde{\gamma}(t)$  intersects  $M_1 \times \tilde{M}_\star$ . Now, we can show the following kind of isoperimetric inequality. This inequality concludes the proof.

$$\begin{aligned}
\left| \int_{D^2} \tilde{\gamma}^* \pi_2^* \omega_2 \right| &= \left| \int_{D^2} (\tilde{\gamma}_2)^* \tilde{\omega}_2 \right| = \left| \int_0^1 \tilde{\gamma}_2^* \lambda_{\tilde{M}_2} \right| \\
&\leq \|\lambda_{\tilde{M}_2}|_{\gamma_2(S^1)}\|_{L^\infty} \int_0^1 \|\partial_t \tilde{\gamma}_2\|_{\tilde{g}_2} dt \\
&= \|\lambda_{\tilde{M}_2}|_{\gamma_2(S^1)}\|_{L^\infty} \int_0^1 \|\partial_t \gamma_2\|_{g_2} dt \\
&= \|\lambda_{\tilde{M}_2}|_{\gamma_2(S^1)}\|_{L^\infty} \int_0^1 \|J \partial_s \gamma_2 + \pi_{2*} X_F(t, \gamma_2)\|_{g_2} dt \\
&\leq \lambda_{\text{Max}} (\|\nabla_m \mathcal{A}_F^{\tilde{H}}(v(s_0, \cdot), \eta(s_0))\|_m + \|X_F\|_{L^\infty}).
\end{aligned}$$

where

$$\begin{aligned}
\lambda_{\text{Max}} &:= \max_{x \in \tilde{M}_2} \left\{ \|\lambda_{\tilde{M}_2}(x)\|_{\tilde{g}_2} \mid d_{\tilde{g}_2}(x, \tilde{M}_\star) < \int_0^1 \|\partial_t \gamma_2\|_{g_2} dt \right\} \\
&\leq \max_{x \in \tilde{M}_2} \left\{ \|\lambda_{\tilde{M}_2}(x)\|_{\tilde{g}_2} \mid d_{\tilde{g}_2}(x, \tilde{M}_\star) < \|\nabla_m \mathcal{A}_F^{\tilde{H}}(v(s_0, \cdot), \eta(s_0))\|_m + \|X_F\|_{L^\infty} \right\}.
\end{aligned}$$

□

The following two lemmas can be proved similarly to the corresponding lemmas in the previous section.

**Lemma 3.3.** We assume that for  $(v, \eta) \in C^\infty(S^1, M_1 \times M_2) \times \mathbb{R}$ ,  $v(t) \in U_\delta := \tilde{G}^{-1}(-\delta, \delta)$  for all  $t \in (\frac{1}{2}, 1)$  with  $0 < 2\delta < \min\{1, \delta_0\}$ . Then there exists  $C > 0$  satisfying

$$|\eta| \leq C \left( |\mathcal{A}_F^{\tilde{H}}(v, \eta)| + \|\nabla_m \mathcal{A}_F^{\tilde{H}}(v, \eta)\|_m + \left| \int_{D^2} \tilde{v}^* \pi_2^* \omega_2 \right| + 1 \right).$$

**Lemma 3.4.** For  $(v, \eta) \in C^\infty(S^1, M_1 \times M_2) \times \mathbb{R}$  if there exists  $t \in [\frac{1}{2}, 1]$  such that  $v(t) \notin U_\delta$ , then  $\|\nabla_m \mathcal{A}_F^{\tilde{H}}(v, \eta)\|_m > \epsilon$  for some  $\epsilon = \epsilon_\delta$ .

Due to the three previous lemmata, we deduce the fundamental lemma in the situation of Theorem B, and thus we obtain a uniform  $L^\infty$ -bound on the Lagrange multiplier  $\eta$  by the same argument as in the previous section.

**Lemma 3.5.** For a gradient flow line  $w(s) = (v, \eta)(s) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R})$ , the following assertions holds with some  $C, \epsilon > 0$ . If  $\|\nabla_m \mathcal{A}_F^{\tilde{H}}(v, \eta)\|_m < \epsilon$ ,

$$|\eta| \leq C (|\mathcal{A}_F^{\tilde{H}}(w_-)| + |\mathcal{A}_F^{\tilde{H}}(w_+)| + \epsilon + \Xi_\epsilon + 1) \quad \text{provided that} \quad \|\nabla_m \mathcal{A}_F^{\tilde{H}}(v, \eta)\|_m < \epsilon$$

where  $\Xi_\epsilon = \max\{\|\lambda_{\tilde{M}_2}(x)\|_{\tilde{g}_2} \mid d_{\tilde{g}_2}(x, \tilde{M}_\star) < \epsilon + \|X_F\|_{L^\infty}\} (\epsilon + \|X_F\|_{L^\infty}) < \infty$ .

PROOF. The proof is almost same as the proof of Lemma 2.13. Since  $\|\nabla_m \mathcal{A}_F^{\tilde{H}}(v, \eta)\|_m < \epsilon$ ,  $v(t) \subset U_\delta$  for  $t \in (\frac{1}{2}, 1)$  by Lemma 3.4. Thus Lemma 3.2 and Lemma 3.3 prove the lemma. □

This fundamental lemma proves compactness of gradient flow lines as before. Let  $\phi \in \text{Ham}_c(M_1 \times M_2, \omega_1 \oplus \omega_2)$  be a Hamiltonian diffeomorphism with the Hofer norm less than  $\wp(\Sigma_1, \lambda_1)$ . We consider a moduli space of gradient flow lines of the Rabinowitz action functional perturbed by a special smooth family of Hamiltonian functions. Then in the boundary of this moduli space, there is a broken gradient flow line of which one asymptotic end gives rise to either a leafwise intersection point of  $\phi$  or a closed Reeb orbit with period less than  $\|\phi\|$ . But since  $\|\phi\| < \wp(\Sigma_1, \lambda_1)$ , there is no such a closed Reeb orbit and hence we obtain a leafwise intersection point. This is so called the *stretching the neck* argument, see [1, 24]. Even further, there exists a leafwise intersection point even if  $\Sigma_1 \times M_2$  does not bound a compact region in  $M_1 \times M_2$  due to the arguments in [23, 24]. Next, we define the Rabinowitz Floer homology for  $(\Sigma_1 \times M_2, M_1 \times M_2)$  in the same way as before and derive the Künneth formula in this situation. First of all, we consider another two action functionals  $\mathcal{A}^H : \mathcal{L}_{M_1} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\mathcal{A} : \mathcal{L}_{M_2} \rightarrow \mathbb{R}$  defined by

$$\mathcal{A}^H(v_1, \eta) := - \int_0^1 v_1^* \lambda_1 - \eta \int_0^1 H(t, v) dt, \quad \mathcal{A}(v_2) := - \int_{D^2} \bar{v}_2^* \omega_2.$$

where  $H(t, x) = \chi(t)G(x) \in C^\infty(S^1 \times M_1)$ . We note that  $\mathcal{A}^{\tilde{H}}$  is defined on  $\mathcal{L}_{M_1 \times M_2} \times \mathbb{R}$ . As in the proof of Theorem A, we compare critical points of  $\mathcal{A}^{\tilde{H}}$  and  $\mathcal{A}^H$  as follows.

$$\text{Crit}_n(\mathcal{A}^{\tilde{H}}) = \bigcup_{i+j=n} \text{Crit}_i(\mathcal{A}^H) \times \text{Crit}_j(\mathcal{A}).$$

Since  $\text{Crit} \mathcal{A}$  consists of one component  $M_2$ , any gradient flow line with cascades of  $\mathcal{A}$  necessarily has zero cascades, and hence is simply a gradient flow line of an additional Morse function  $f \in C^\infty(M_2)$ . Thus the chain group for the Morse-Bott homology of  $\mathcal{A}$  is given by  $\text{CF}(\mathcal{A}, f) = \text{CM}(f)$ . Here CM stands for the Morse complex. The following map is a chain isomorphism, which can be verified using the methods of the previous subsection.

$$P_n : (\text{CF}_*(\mathcal{A}^H) \otimes \text{CM}_*(f))_n \longrightarrow \text{CF}_n(\mathcal{A}^{\tilde{H}}), \\ (v_1, \eta) \otimes v_2 \longmapsto ((v_1, v_2), \eta).$$

Therefore it induces an isomorphism on the homology level:

$$(P_\bullet)_* : \text{H}_\bullet(\text{CF}(\mathcal{A}^H) \otimes \text{CM}(f)) \xrightarrow{\cong} \text{H}_\bullet(\text{CF}(\mathcal{A}^{\tilde{H}})) = \text{RFH}_\bullet(\Sigma_1 \times M_2, M_1 \times M_2).$$

Finally, the Künneth formula for  $(\Sigma_1 \times M_2, M_1 \times M_2)$  directly follows:

$$\text{RFH}_n(\Sigma_1 \times M_2, M_1 \times M_2) \cong \bigoplus_{p=0}^n \text{RFH}_p(\Sigma_1, M_1) \otimes \text{H}_{n-p}(M_2).$$

#### 4. APPLICATIONS

As we have mentioned in the introduction, we cannot achieve compactness of gradient flow lines of  $\mathcal{A}_F^{\tilde{H}_1, \tilde{H}_2}$  for an arbitrary perturbation  $F$ . For that reason, the existence problem of leafwise intersection points for a product submanifold which is not of contact type is still open. On the other hand, the existence of leafwise intersection points for contact coisotropic submanifolds was already proved in [21, 24]. Furthermore, due to the Künneth formula, we can deduce the existence of infinitely many leafwise intersection points for some kind of product submanifolds of contact type. First, we recall the notion of contact condition on coisotropic submanifolds introduced by Bolle [10, 11].



**Definition 4.1.** A coisotropic submanifold  $\Sigma$  of codimension  $k$  in a symplectic manifold  $(M, \omega)$  is called of *restricted contact type* if there exist global one forms  $\lambda_1, \dots, \lambda_k \in \Omega^1(M)$  which satisfy

- (1)  $d\lambda_i = \omega$  for  $i = 1, \dots, k$ ;
- (2)  $\lambda_1 \wedge \dots \wedge \lambda_k \wedge \omega^{n-k}|_{\Sigma} \neq 0$ .

**Remark 4.2.** [11, 20] Let  $\Sigma$  be closed and have contact type in  $M$ . Then a one form  $\lambda = a_1\lambda_1 + \dots + a_k\lambda_k$  with  $a_1 + \dots + a_k = 0$  is closed and hence defines an element of  $H_{\text{dR}}^1(\Sigma)$ . In addition,  $\lambda \neq 0$  is not exact; otherwise  $\lambda = df$  for some  $f \in C^\infty(\Sigma)$ , and hence  $\lambda(x) = 0$  at a critical point  $x$  of  $f$ , but condition (ii) yields that  $\lambda_1, \dots, \lambda_k$  are linearly independent on  $\Sigma$ ; thus  $\lambda_1(x) = \dots = \lambda_k(x) = 0$ . As a result,  $\dim H_{\text{dR}}^1(\Sigma) \geq k - 1$ . It imposes restrictions on the contact condition; for instance,  $S^3 \times S^3$  is not of contact type in  $\mathbb{R}^8$ .

We note that if the codimension of  $\Sigma$  is bigger than one,  $\Sigma$  never bounds a compact region in  $M$ . In spite of such a dimension problem, the condition that global coordinates exist (roughly speaking, Poisson-commuting Hamiltonian functions whose common zero locus is only  $\Sigma$ ) enable us to unfold the generalized Rabinowitz Floer homology theory [24]. It turns out that a product of contact hypersurfaces bounding respective ambient symplectic manifolds has global coordinates.

**Theorem 4.3.** [24] *If  $\Sigma$  is a contact coisotropic submanifold of  $M$  which admits global coordinates, then the Floer homology of the perturbed Rabinowitz action functional is well-defined.*

Since the Rabinowitz action functional can be defined for each homotopy classes of loops, we can define the Rabinowitz Floer homology  $\text{RFH}(\Sigma, M, \gamma)$  for  $\gamma \in [S^1, M]$ . We note that the  $\text{RFH}(\Sigma, M)$  considered so far, is equal to  $\text{RFH}(\Sigma, M, \text{pt})$ . Moreover we also define Rabinowitz Floer homology on the full loop space and denote it by  $\mathbf{RFH}(\Sigma, M)$ . Then we have

$$\mathbf{RFH}_*(\Sigma, M) = \bigoplus_{\gamma \in [S^1, M]} \text{RFH}_*(\Sigma, M, \gamma).$$

We recall the computation of Rabinowitz Floer homology on the (unit) cotangent bundle  $(S^*N, T^*N)$  for a closed Riemannian manifold  $N$ .

**Theorem 4.4.** [14, 8, 25]

$$\mathbf{RFH}_*(S^*N, T^*N) \cong \begin{cases} H_*(\Lambda N), & * > 1, \\ H^{-*+1}(\Lambda N), & * < 0. \end{cases}$$

Here  $\Lambda N$  stands for the free loop space of  $N$ .

Since the Künneth formula obviously holds for  $\mathbf{RFH}$  also, the following corollary directly follows.

**Corollary 4.5.** *If  $\mathbf{RFH}_*(\Sigma_1, M_1) \neq 0$ , and  $\dim H_*(\Lambda N) = \infty$  then*

$$\dim \mathbf{RFH}_*(\Sigma_1 \times S^*N, M_1 \times T^*N) = \infty.$$

Accordingly, if  $\Sigma_1 \times S^*N$  has contact type again,  $\Sigma_1 \times S^*N$  has infinitely many leafwise intersection points or a periodic leafwise intersections for a generic perturbation.

**4.1. Proof of Corollary A and B.** From now on, we investigate leafwise intersections on  $(S^*S^1 \times S^*N, T^*S^1 \times T^*N)$ .

**Lemma 4.6.**  $S^*S^1 \times S^*N$  is a contact submanifold of codimension two in  $T^*S^1 \times T^*N$ .

PROOF.  $(T^*S^1, \omega_{S^1, \text{can}}) \cong (S^1 \times \mathbb{R}, d\theta \wedge dr)$  where  $\theta$  is the angular coordinate on  $S^1$  and  $r$  is the coordinate on  $\mathbb{R}$ . Then  $d\theta \wedge dr$  has two global primitives  $-rd\theta$  and  $-rd\theta + d\theta$ . We can easily check that  $S^*S^1 \times S^*N$  carries a contact structure with  $-rd\theta \oplus \lambda_{N, \text{can}}$  and  $(-rd\theta + d\theta) \oplus \lambda_{N, \text{can}}$  where  $\lambda_{N, \text{can}}$  is the canonical one form on  $T^*N$ .  $\square$

To exclude periodic leafwise intersections, we consider the loop space  $\Omega$  defined by

$$\Omega := \{v = (v_1, v_2) \in C^\infty(S^1, T^*S^1 \times T^*N) \mid v_1 \text{ is contractible in } T^*S^1\}.$$

Then we define the Rabinowitz action functional on this loop space,  $\mathcal{A}_F^{H_1, H_2} : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ , and construct the respective Rabinowitz Floer homology  $\text{RFH}(S^*S^1 \times S^*N, T^*S^1 \times T^*N, \Omega)$  as before. Moreover the following type of the Künneth formula holds.

$$\text{RFH}_n(S^*S^1 \times S^*N, T^*S^1 \times T^*N, \Omega) \cong \bigoplus_{p=0}^n \text{RFH}_p(S^*S^1, T^*S^1) \otimes \mathbf{RFH}_{n-p}(S^*N, T^*N).$$

Therefore  $\text{RFH}(S^*S^1 \times S^*N, T^*S^1 \times T^*N, \Omega)$  is of infinite dimensional if  $\dim H_*(\Lambda N) = \infty$  and Lemma 4.7 below yields that there are infinitely many leafwise intersection points for a generic perturbation whenever  $\dim N \geq 2$ . This proves Corollary A.

In order to prove that there is generically no periodic leafwise intersections, we review the argument in [2]. We denote by  $\mathcal{R}$  the set of closed Reeb orbits in  $T^*N$  which has dimension one. It is convenient to introduce the following sets:

$$\mathcal{F}^j = \{F \in C_c^j(S^1 \times T^*S^1 \times T^*N) \mid F(t, \cdot) = 0, \forall t \in [\frac{1}{2}, 1]\}, \quad \mathcal{F} = \bigcap_{j=1}^{\infty} \mathcal{F}^j.$$

**Lemma 4.7.** If  $\dim N \geq 2$ , then the set

$$\mathcal{F}_{S^*S^1 \times S^*N} := \left\{ F \in \mathcal{F} \mid \begin{array}{l} \mathcal{A}_F^{H_1, H_2} \text{ is Morse \& } v(0) \cap (S^*S^1 \times R) = \emptyset \\ \text{for all } \forall (v, \eta_1, \eta_2) \in \mathcal{A}_F^{H_1, H_2}, R \in \mathcal{R} \end{array} \right\}$$

is dense in the set  $\mathcal{F}$ .

PROOF. In this proof, we denote by

$$\Omega^{1,2} := \{v = (v_1, v_2) \in W^{1,2}(S^1, T^*S^1 \times T^*N) \mid v_1 \text{ is contractible in } T^*S^1\}.$$

the loop space which is indeed a Hilbert manifold. Let  $\mathcal{E}$  be the  $L^2$ -bundle over  $\Omega^{1,2}$  with  $\mathcal{E}_v = L^2(S^1, v^*T(S^*S^1 \times S^*N))$ . We consider the section

$$S : \Omega^{1,2} \times \mathbb{R}^2 \times \mathcal{F}^j \rightarrow \mathcal{E}^\vee \times \mathbb{R}^2 \quad \text{defined by} \quad S(v, \eta_1, \eta_2, F) := d\mathcal{A}_F^{H_1, H_2}(v, \eta_1, \eta_2).$$

Here the symbol  $\vee$  represents the dual space. At  $(v, \eta_1, \eta_2, F) \in S^{-1}(0)$ , the vertical differential

$$DS : T_{(v, \eta_1, \eta_2, F)} \Omega^{1,2} \times \mathbb{R}^2 \times \mathcal{F}^j \rightarrow \mathcal{E}_v^\vee \times \mathbb{R}^2$$

is given by the pairing

$$\langle DS_{(v, \eta_1, \eta_2, F)}[\hat{v}^1, \hat{\eta}_1^1, \hat{\eta}_2^1, \hat{F}], [\hat{v}^2, \hat{\eta}_1^2, \hat{\eta}_2^2] \rangle = \mathcal{H}_{\mathcal{A}_F^{H_1, H_2}}[(\hat{v}^1, \hat{\eta}_1^1, \hat{\eta}_2^1), (\hat{v}^2, \hat{\eta}_1^2, \hat{\eta}_2^2)] + \int_0^1 \hat{F}(t, v) dt.$$

where  $\mathcal{H}_{\mathcal{A}_F^{H_1, H_2}}$  is the Hessian of  $\mathcal{A}_F^{H_1, H_2}$ . Due to the arguments in [1] (in fact they proved the surjectivity for  $\mathcal{A}_F^H$ , but their proof obviously can be extended to our situation, see also [24]), we know that for  $(v, \eta_1, \eta_2, F) \in S^{-1}(0)$ ,  $DS_{(v, \eta_1, \eta_2, F)}$  is surjective on the space

$$\mathcal{V} := \{(\hat{v}, \hat{\eta}_1, \hat{\eta}_2, \hat{F}) \in T_{(v, \eta_1, \eta_2, F)}(\Omega^{1,2} \times \mathbb{R}^2 \times \mathcal{F}^j) \mid \hat{v}(0) = 0\}.$$

Next, we consider the evaluation map

$$\begin{aligned} \text{ev} : \mathcal{M} &\longrightarrow S^*S^1 \times S^*N, \\ (v, \eta_1, \eta_2, F) &\longmapsto v(0). \end{aligned}$$

Since  $DS_{(v, \eta_1, \eta_2, F)}|_{\mathcal{V}}$  is surjective, Lemma 4.8 below implies that  $\text{ev}$  is a submersion. Then  $\mathcal{M}_{\mathcal{R}} := \text{ev}^{-1}(S^*S^1 \times \mathcal{R})$  is a submanifold in  $\mathcal{M}$  of

$$\text{codim}(\mathcal{M}_{\mathcal{R}}/\mathcal{M}) = \text{codim}(S^*S^1 \times \mathcal{R}/S^*S^1 \times S^*N).$$

We consider the projections  $\Pi : \mathcal{M} \longrightarrow \mathcal{F}^j$  and  $\Pi_{\mathcal{R}} := \Pi|_{\mathcal{M}_{\mathcal{R}}}$ . Then  $\mathcal{A}_F^{H_1, H_2}$  is Morse if and only if  $F$  is a regular value of  $\Pi$ , which is a generic property by Sard-Smale theorem (for  $j$  large enough). The set  $\Pi^{-1}(F)$  of leafwise intersection points for  $F$  is manifold of required dimension zero since it is a critical set of  $\mathcal{A}_F^{H_1, H_2}$ . On the other hand,  $\Pi_{\mathcal{R}}^{-1}(F)$  is a manifold of dimension

$$0 + \dim \mathcal{M}_{\mathcal{R}} - \dim \mathcal{M} = -\text{codim}(\mathcal{M}_{\mathcal{R}}/\mathcal{M}) < 0$$

since we have assumed  $\dim N \geq 2$ . Therefore  $\text{ev}$  does not intersect  $S^*S^1 \times \mathcal{R}$ , so the set

$$\mathcal{F}_{S^*S^1 \times S^*N}^j := \mathcal{F}_{S^*S^1 \times S^*N} \cap \mathcal{F}^j$$

is dense in  $\mathcal{F}$  for all  $j \in \mathbb{N}$ . Since  $\mathcal{F}_{S^*S^1 \times S^*N}$  is the countable intersection of  $\mathcal{F}_{S^*S^1 \times S^*N}^j$  for  $j \in \mathbb{N}$ , it is dense again in  $\mathcal{F}$  and the lemma is proved.  $\square$

**Lemma 4.8. (Salamon)** Let  $\mathcal{E} \longrightarrow \mathcal{B}$  be a Banach bundle and  $s : \mathcal{B} \longrightarrow \mathcal{E}$  a smooth section. Moreover, let  $\phi : \mathcal{B} \longrightarrow N$  be a smooth map into the Banach manifold  $N$ . We fix a point  $x \in s^{-1}(0) \subset \mathcal{B}$  and set  $K := \ker d\phi(x) \subset T_x\mathcal{B}$  and assume the following two conditions.

- (1) The vertical differential  $Ds|_K : K \longrightarrow \mathcal{E}_x$  is surjective.
- (2)  $d\phi(x) : T_x\mathcal{B} \longrightarrow T_{\phi(x)}N$  is surjective.

Then  $d\phi(x)|_{\ker Ds(x)} : \ker Ds(x) \longrightarrow T_{\phi(x)}N$  is surjective.

PROOF. Given  $\xi \in T_{\phi(x)}N$ , condition (ii) implies that there exists  $\eta \in T_x\mathcal{B}$  satisfying  $d\phi(x)\eta = \xi$ . In addition, by condition (i), there exists  $\zeta \in K \subset T_x\mathcal{B}$  satisfying  $Ds(x)\zeta = Ds(x)\eta$ . We set  $\tau := \eta - \zeta$  and compute

$$Ds(x)\tau = Ds(x)\eta - Ds(x)\zeta = 0$$

thus,  $\tau \in \ker Ds(x)$ . Moreover,

$$d\phi(x)\tau = d\phi(x)\eta - \underbrace{d\phi(x)\zeta}_{=0} = d\phi(x)\eta = \xi$$

proves the lemma.  $\square$

In the case of Theorem B, we also redefine the Rabinowitz action functional  $\mathcal{A}_F^H : \Omega_{M_2} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$\mathcal{A}_F^H(v, \eta) = - \int_0^1 v_1^* \lambda_1 - \int_{D^2} \bar{v}_2^* \omega_2 - \eta \int_0^1 H(t, v) dt - \int_0^1 F(t, v) dt$$

where

$$\Omega_{M_2} : \{v = (v_1, v_2) \in C^\infty(S^1, M_1 \times M_2) \mid v_2 \text{ is contractible in } M_2\}.$$

We can also define the respective Rabinowitz Floer homology and derive an appropriate Künneth formula as before.

**Corollary 4.9.** *Let  $(M_2, \omega_2)$  be a closed, symplectically aspherical symplectic manifold. If a closed manifold  $N$  has  $\dim H_*(\Lambda N) = \infty$ , then*

$$\dim \text{RFH}_*(S^*N \times M_2, T^*N \times M_2, \Omega_{M_2}) = \infty.$$

*Therefore, if  $\dim N \geq 2$ ,  $S^*N \times M_2$  has infinitely many leafwise intersection points for a generic perturbation.*

The previous corollary proves Corollary B.

**Remark 4.10.** The corollaries still holds when we deal with a generic fiber-wise star shaped hypersurface  $\Sigma \subset T^*N$  instead of  $S^*N$ , see [2].

## 5. APPENDIX : INDEX FOR RABINOWITZ FLOER HOMOLOGY

In fact, we are able to derive the Künneth formula and obtain applications without defining indices. Nevertheless, for the sake of completeness, we briefly recall the index for generators of the Rabinowitz Floer chain complex in this appendix (see [13] for the detailed arguments). Let  $\Sigma$  be a contact hypersurface in  $M$ . Under the following assumption the Rabinowitz Floer homology has  $\mathbb{Z}$ -grading,

(H1) Closed Reeb orbits on  $(\Sigma, \lambda)$  is of Morse-Bott type [13].

(H2) The first chern class  $c_1$  vanishes on  $TM$ .

**Remark 5.1.** Without any hypothesis on the first chern class, the Rabinowitz Floer homology has  $\mathbb{Z}/2$ -grading. The non-degeneracy assumption (H1) is satisfied for generic hypersurfaces and the invariance property allows us to perturb a hypersurface to be Morse-Bott type.

Let  $\mathcal{M}$  be the moduli space of all finite energy gradient flow lines of  $\mathcal{A}^H$  and  $w = (v, \eta) \in C^\infty(\mathbb{R} \times S^1, M) \times C^\infty(\mathbb{R}, \mathbb{R})$  be a gradient flow line of  $\mathcal{A}^H$  with  $\lim_{s \rightarrow \pm\infty} w(s) = w_\pm = (v_\pm, \eta_\pm) \in \text{Crit} f$  and  $v_\pm \subset C_\pm$  where  $C_\pm \subset \text{Crit} \mathcal{A}^H$  are connected components of the critical manifold and  $f$  is an additional Morse function on a critical manifold  $\text{Crit} \mathcal{A}^H$ . The linearization of the gradient flow equation along  $(v, \eta)$  gives rise to an operator  $D_{(v, \eta)}^{\mathcal{A}^H}$ . For suitable weighted Sobolev spaces,  $D_{(v, \eta)}^{\mathcal{A}^H}$  is a Fredholm operator. Then the local virtual dimension of  $\mathcal{M}$  at  $(v, \eta)$  is defined to be

$$\text{virdim}_{(v, \eta)} \mathcal{M} := \text{ind} D_{(v, \eta)}^{\mathcal{A}^H} + \dim C_- + \dim C_+.$$

Here  $\text{ind} D_{(v, \eta)}^{\mathcal{A}^H}$  stands for the Fredholm index of  $D_{(v, \eta)}^{\mathcal{A}^H}$ . Cieliebak-Frauenfelder [13] investigated the spectral flow of the Hessian  $\text{Hess}_{\mathcal{A}^H}$  and consequently proved the following index formula:

$$\text{virdim}_{(v, \eta)} \mathcal{M} = \mu_{\text{CZ}}(v_+) - \mu_{\text{CZ}}(v_-) + \frac{\dim C_- + \dim C_+}{2}.$$

Here  $\mu_{\text{CZ}}$  is the *Conley-Zehnder index* defined below. Since a closed Reeb orbit  $v_+$  is contractible in  $M$ , we have a filling disk  $\bar{v}_+ : D^2 \rightarrow M$  such that  $\bar{v}_+|_{\partial D^2} = v_+$ . The filling disk  $\bar{v}_+$  determines homotopy class of trivialization of the symplectic vector bundle  $(\bar{v}_+)^* TM$ . The linearized flow of the Reeb vector field along  $v_+$  defines a path in  $Sp(2n, \mathbb{R})$  the group of

symplectic matrices. The Conley Zehnder index of  $v_+$  is defined by the Maslov index of [26] this path. This index is independent of choice of filling disk due to (H2) because the Conley Zehnder indices of different filling disks differ by  $c_1$ . In the same way  $\mu_{\text{CZ}}(v_-)$  is also defined. Now, we are in a position to define a grading  $\mu_{\text{RFH}}$  on  $\text{CF}(\mathcal{A}^H)$  by

$$\mu_{\text{RFH}}(v_{\pm}, \eta_{\pm}) := \mu_{\text{CZ}}(v_{\pm}) + \mu_{\sigma}^f(v_{\pm}).$$

where  $\mu_{\sigma}^f$  is the *signature index* defined by

$$\begin{aligned} \mu_{\sigma}^f(v_{\pm}) &= -\frac{1}{2} \text{sign}(\text{Hess}_f(v_{\pm})) \\ &= -\frac{1}{2} \left( \# \left\{ \begin{array}{l} \text{positive eigenvalues of} \\ \text{the Hessian of } f \text{ at } v_{\pm} \end{array} \right\} - \# \left\{ \begin{array}{l} \text{negative eigenvalues of} \\ \text{the Hessian of } f \text{ at } v_{\pm} \end{array} \right\} \right). \end{aligned}$$

We note that by definition,

$$\mu_{\sigma}^f(v_{\pm}) = \mu_f^{\text{Morse}}(v_{\pm}) - \frac{1}{2} \dim C_{\pm}.$$

Then we notice that the dimension of gradient flow lines of  $\mathcal{A}_F^H$  interchanging  $w_-$  and  $w_+$  equals the index difference of the two critical points by the following computation.

$$\begin{aligned} \dim \widehat{\mathcal{M}}\{w_-, w_+\} &= \text{vir} \dim_{(v, \eta)} \mathcal{M} - \dim C_+ - \dim C_- + \dim W_f^u(v_-) + \dim W_f^s(v_+) \\ &= \mu_{\text{CZ}}(v_-) - \mu_{\text{CZ}}(v_+) - \frac{\dim C_- + \dim C_+}{2} + \mu_f^{\text{Morse}}(w_-) + \dim C^+ - \mu_f^{\text{Morse}}(w_+) \\ &= \mu_{\text{CZ}}(v_-) - \mu_{\text{CZ}}(v_+) - \frac{\dim C_- + \dim C_+}{2} + \mu_{\sigma}^f(v_-) + \frac{1}{2} \dim C_- + \dim C_+ \\ &\quad - (\mu_{\sigma}^f(v_+) + \frac{1}{2} \dim C_+) \\ &= \mu_{\text{CZ}}(v_-) - \mu_{\text{CZ}}(v_+) + \mu_{\sigma}^f(v_-) - \mu_{\sigma}^f(v_+) \\ &= \mu_{\text{RFH}}(v_-) - \mu_{\text{RFH}}(v_+) \end{aligned} \tag{5.1}$$

where  $W_f^s(v^+)(W_f^u(v^-))$  is the (un)stable manifold with respect to  $(f, v^{\pm})$ .

Furthermore, the RFH-index of  $((v_1, v_2), \eta_1, \eta_2) \in \text{Crit} \mathcal{A}^{\tilde{H}_1, \tilde{H}_2}$  (as used in Theorem A) splits into the indices of  $(v_1, \eta_1)$  and  $(v_2, \eta_2)$ .

$$\mu_{\text{RFH}}((v_1, v_2), \eta_1, \eta_2) = \mu_{\text{RFH}}(v_1, \eta_1) + \mu_{\text{RFH}}(v_2, \eta_2) \tag{5.2}$$

since the Conley-Zehnder index (in fact, the Maslov index) and the Morse index behave additively under the direct sum operation.

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